

# BOUNDS FOR INCREASING SEQUENCES WITH DISTINCT CONSECUTIVE SUMS (ERDŐS PROBLEM 357)

CHANDLER SNIVELY

ABSTRACT. Let  $f(n)$  be the largest  $k$  for which there are integers  $1 \leq a_1 < \dots < a_k \leq n$  such that the sums  $a_i + a_{i+1} + \dots + a_{i'}$  ( $1 \leq i \leq i' \leq k$ ) are pairwise distinct. Whether  $f(n) = o(n)$  is open (Erdős Problem 357). We prove

$$f(n) \leq \frac{44273}{90000}n + 2\sqrt{n} + 177792 \quad \text{for all } n \geq 0,$$

so that  $\limsup_{n \rightarrow \infty} f(n)/n \leq \frac{44273}{90000} = \frac{1}{2} - \frac{2908}{360000}$ ; the upper bounds on  $f$  that we found in the literature all have leading term at least  $n/2$ . We also prove

$$f(n) \geq \frac{15}{\sqrt{21}}\sqrt{n} - 1800 \quad \text{for all } n \geq 12096 \cdot 504^2,$$

so that  $\liminf_{n \rightarrow \infty} f(n)/\sqrt{n} \geq 15/\sqrt{21} = 3.27326\dots$ ; the best lower bound we found in the literature is  $(4/\sqrt{3} - o(1))\sqrt{n}$ . The upper bound combines a crossing argument with a lemma that lengthens short blocks by one term, and forces many integers of  $(n/2, 2n/3]$  to be missing from the sequence by using pairs of runs of consecutive integers with equal sums. The lower bound uses a periodic pattern of residues modulo 42 with one added jump, which reduces every possible coincidence of block sums to a divisor equation that a counting argument avoids. The paper gives complete written proofs of these bounds. They are also formally verified in Lean 4 with Mathlib, for a function  $f$  that is checked in Lean to be definitionally equal to the one defined in the Formal Conjectures project; they do not settle the open question whether  $f(n) = o(n)$ .

## 1. INTRODUCTION

Call a finite sequence of positive integers  $a_1, \dots, a_k$  *valid* if its consecutive sums

$$a_i + a_{i+1} + \dots + a_{i'} \quad (1 \leq i \leq i' \leq k)$$

are pairwise distinct. Following the Erdős problems database [3], let

$$f(n) = \max\{k : \text{there is a valid sequence } 1 \leq a_1 < a_2 < \dots < a_k \leq n\}.$$

Problem 357 of [3] asks how  $f(n)$  grows, and in particular whether  $f(n) = o(n)$ . Erdős and Graham [13, p. 58] record this question of Erdős and Harzheim, together with the variants without monotonicity or without distinctness; Erdős [12, p. 70] discussed the analogous question for infinite sequences. Equivalently, the partial sums  $0 = s_0 < s_1 < \dots < s_k$ ,  $s_i = a_1 + \dots + a_i$ , form a Sidon set whose consecutive differences increase and are at most  $n$ . We do not decide whether  $f(n) = o(n)$ ; we prove explicit bounds on both sides.

**Theorem 1.1** (Upper bound). *For every integer  $n \geq 0$ ,*

$$720000 f(n) \leq 354184n + 1440000 \lfloor \sqrt{n} \rfloor + 128010240000,$$

---

*Date:* 23 October 2026.

*2020 Mathematics Subject Classification.* 11B13, 11B75, 05D05.

*Key words and phrases.* distinct consecutive sums, block sums, Sidon sets, Erdős problems, formal verification, Lean.

and consequently

$$f(n) \leq \frac{44273}{90000}n + 2\sqrt{n} + 177792, \quad \frac{44273}{90000} = \frac{1}{2} - \frac{2908}{360000} = 0.4919\bar{2}.$$

Because of its large additive constant, the right-hand side is below  $n/2$  exactly for  $n \geq 23202647$ . The same method with a simpler design gives  $f(n) \leq \frac{797}{1620}n + 2\sqrt{n} + 7$  for all  $n$  (Remark 4.9; it is proved in Lean, and we give only an outline); its right-hand side is below  $n/2$  exactly for  $n \geq 63849$ , and it is the smaller of the two bounds for every  $n \leq 3348973255$ .

**Theorem 1.2** (Lower bound, modulus 42). *For every integer  $m \geq 0$  we have  $f(21m^2 + 105m + 46) \geq 12m$  (for  $m = 0$  the empty sequence). Consequently, for every  $n \geq 0$ ,*

$$f(n) \geq \frac{12}{\sqrt{21}}\sqrt{n} - 48, \quad \frac{12}{\sqrt{21}} = 2.618614\dots$$

**Theorem 1.3** (Lower bound, with one skip). *For every integer  $j \geq 500$  we have  $f(12096j^2 + 87024j + 81) \geq 360j$ . Consequently*

$$\begin{aligned} f(n) &\geq \frac{15}{\sqrt{21}}\sqrt{n} - 1800 && \text{for } n \geq 12096 \cdot 504^2 = 3,072,577,536, \\ f(n) &\geq \frac{15}{\sqrt{21}}\sqrt{n} - 181440 && \text{for all } n \geq 0, \end{aligned}$$

where  $\frac{15}{\sqrt{21}} = \frac{360}{\sqrt{12096}} = 3.273268\dots$

Here  $12096 = 21 \cdot 24^2$ , so that  $360j/\sqrt{12096j^2} = 15/\sqrt{21}$ . The proofs give the slightly smaller arguments  $21m^2 + 105m + 34$  and  $12096j^2 + 87024j + 76$ ; the forms stated are the ones verified in Lean. The decimal expansions above are truncated, not rounded; the exact constants are the fractions and radicals.

In particular

$$\liminf_{n \rightarrow \infty} \frac{f(n)}{\sqrt{n}} \geq \frac{15}{\sqrt{21}} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{f(n)}{n} \leq \frac{44273}{90000}.$$

The gap between  $\sqrt{n}$  and  $n$  is untouched: nothing in this paper bears on whether  $f(n) = o(n)$ .

**Formal verification.** All statements of Theorems 1.1–1.3, and the variant in Remark 4.9, are formally verified in the Lean 4 proof assistant [7] with the Mathlib library [23], against a definition of  $f$  that is definitionally equal to the one in the Formal Conjectures project [14] (Section 7). The Lean files, the scripts that repeat every check, and the programs and data of Section 8 are in the accompanying repository [21]. The written proofs of Theorems 1.1–1.3 in Sections 3–6 are complete without the formalization; Remark 4.9 is proved only in Lean. The formal proofs follow the same arguments, so they check those arguments mechanically; they are not independent proofs.

**How the results were obtained.** The constructions, the proofs and their formalizations were produced by AI systems, under the direction of the author; the statement on AI use at the end gives the details, and Section 9 describes how the ideas arose. We report this because it is relevant to how the results should be checked, and we have arranged the paper so that every step can be checked by hand.

**Organization.** Section 2 reviews previous bounds. Section 3 fixes notation and proves two reductions. Section 4 proves Theorem 1.1; Sections 5 and 6 prove Theorems 1.2 and 1.3. Section 7 describes the formal verification and what it does and does not certify. Section 8 reports finite computations, which are not used in any proof. Section 9 records how the results were found, and Section 10 discusses limitations.

## 2. PREVIOUS RESULTS

Table 1 lists the bounds on  $f$  known to us. Except where marked, we took the bounds from the sources as recorded in [3] and its forum [11], without checking them against the original papers. For the non-monotone analogue  $g(n) \geq f(n)$ , in which the  $a_i$  need not increase, Hegyvári [15] proved  $(\frac{1}{3}+o(1))n \leq g(n) \leq (\frac{2}{3}+o(1))n$  (as recorded in [3], in the zbMATH review of [15], and in [1, 16]); the upper bound applies to  $f$ , the lower bound does not. Coppersmith and Phillips [6] studied the weaker property that no term is the sum of a block of two or more consecutive earlier terms. A valid sequence has this property, since such a term would collide with the block consisting of that term alone, so their upper bound applies to  $f$ ; they also constructed sequences with the weaker property of length  $\frac{13}{24}n - O(1)$ , so no bound below  $\frac{13}{24}n$  can use that property alone (we read only the abstract of [6]).

The lower bound  $(2+o(1))\sqrt{n}$  comes from a stronger property: in an admissible set in the sense of Erdős and Straus, sums of different numbers of distinct elements never coincide, and the largest admissible subset of  $\{1, \dots, N\}$  has  $(2+o(1))\sqrt{N}$  elements [22, 8, 9, 4]. A valid increasing sequence needs the property only for sums of *consecutive* terms; the constructions of [19] and of this paper exploit the difference. For small  $n$  the exact values of  $f$  are known from computer searches: the least  $n$  with  $f(n) \geq k$  is tabulated for  $k \leq 22$  in [20], and  $f(n)$  for  $n \leq 53$  in [5].

To our knowledge, Theorem 1.1 is the first bound of the form  $f(n) \leq (c+o(1))n$  with  $c < \frac{1}{2}$ ; the value  $\frac{1}{2}$  was obtained informally in July 2026 [10, 19, 18]. Also to our knowledge, the constants  $\frac{12}{\sqrt{21}}$  and  $\frac{15}{\sqrt{21}}$ , the constructions of Sections 5–6 and the design of Section 4 have not appeared in the literature; the lower-bound method is Pickhardt’s [19], with a different pattern and an additional skip. These statements rest on the literature searches described in the file `docs/LITERATURE.md` of the repository [21], which is not a proof of novelty. We could not obtain the full texts of [15] and [6]. According to [2], that paper contains a theorem that integer sequences with bounded gaps contain two blocks of different lengths with equal sums; Lemma 4.7(b) below is the simplest instance of this principle, and we do not know whether [15] uses it to bound  $f$  or  $g$ . Hegyvári lectured on consecutive sums, Sidon sets and convex sequences at the CANT conference in 2026; we know only the title and abstract. We cannot exclude unpublished or unindexed work.

## 3. NOTATION AND TWO REDUCTIONS

From now on sequences are indexed from 0. For a finite sequence  $a_0, \dots, a_{k-1}$  and  $0 \leq x \leq y \leq k$  write

$$T(x, y) = a_x + a_{x+1} + \dots + a_{y-1}$$

for the sum of the *block*  $[x, y)$ , so  $T(x, x) = 0$ ; write  $P(y) = T(0, y)$  for the partial sums, and  $S(x, \ell) = T(x, x + \ell)$  for the sum of the block of length  $\ell$  starting at  $x$ . The sequence is valid exactly when  $T(x, y)$ , for  $0 \leq x < y \leq k$ , determines  $(x, y)$ . A valid strictly increasing sequence in  $\{1, \dots, N\}$  is also one in  $\{1, \dots, n\}$  for every  $n \geq N$ , so  $f$  is nondecreasing; and a valid strictly increasing sequence in  $\{1, \dots, n\}$  has at most  $n$  terms, so the maximum defining  $f(n)$  exists.

| Source                                        | Year      | Bound on $f(n)$                                                              | Status                                                                         |
|-----------------------------------------------|-----------|------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| Straus [22], via Weisenberg [11]              | 1966/2025 | $\geq (2 + o(1))\sqrt{n}$                                                    | published construction; observation in a forum comment                         |
| Pickhardt [19]                                | 2026      | $\geq (4/\sqrt{3} - o(1))\sqrt{n}$                                           | web preprint, full text read                                                   |
| This paper, Thm. 1.2                          | 2026      | $\geq \frac{12}{\sqrt{21}}\sqrt{n} - 48$                                     | formally verified                                                              |
| This paper, Thm. 1.3                          | 2026      | $\geq \frac{15}{\sqrt{21}}\sqrt{n} - 1800$<br>( $n \geq 12096 \cdot 504^2$ ) | formally verified                                                              |
| Hegyvári [15] (via $f \leq g$ )               | 1986      | $\leq (\frac{2}{3} + o(1))n$                                                 | published; quoted from [3] and the zbMATH review                               |
| Coppersmith–Phillips [6], via Weisenberg [11] | 1996/2025 | $\leq (\frac{2}{3} - \frac{1}{512})n + O(\log n)$                            | published result; transfer to $f$ in a forum comment                           |
| Beker [1], via Tang [11]                      | 2024/2025 | $\leq (\sqrt{\tanh 1} + o(1))n,$<br>$\sqrt{\tanh 1} = 0.87269\dots$          | published result; deduction in a forum comment; weaker than the two rows above |
| Erdős Problem a Day [10]                      | 2026      | $\leq \frac{n}{2} + O(\sqrt{n})$                                             | informal report; a version of its argument is Corollary 4.2                    |
| Pickhardt [19]; Lenthall-Cleary [18]          | 2026      | $\leq \frac{n}{2} + O(n^{2/3})$                                              | web preprint; unrefereed proof claim                                           |
| This paper, Thm. 1.1                          | 2026      | $\leq \frac{44273}{9000}n + 2\sqrt{n} + 177792$                              | formally verified                                                              |

TABLE 1. Bounds on  $f(n)$  known to us on 23 October 2026.

**Lemma 3.1.** *Let  $0 < a_0 < a_1 < \dots < a_{k-1}$  be integers. If the sequence is not valid, there are  $x, \ell, y, s$  with*

$$1 \leq s < \ell, \quad x + \ell \leq y, \quad y + s \leq k, \quad S(x, \ell) = S(y, s),$$

*that is, an earlier, longer block and a later, shorter, disjoint block with the same sum.*

*Proof.* Take two different nonempty blocks with the same sum. Since the blocks are different, have equal sums, and all terms are positive, neither block contains the other. If they overlap, delete the common part from both: what remains are two nonempty disjoint blocks with equal sums. So we may assume the blocks are disjoint; call them  $(x, \ell)$  and  $(y, s)$  with  $x + \ell \leq y$  (and  $y + s \leq k$ ). If  $s \geq \ell$ , then  $S(y, s) \geq \sum_{i < \ell} a_{y+i} > \sum_{i < \ell} a_{x+i} = S(x, \ell)$ , because  $a_{y+i} > a_{x+i}$ ; hence  $s < \ell$ .  $\square$

The formal statement of the problem (Section 7) phrases validity as injectivity of  $J \mapsto \sum_{i \in J} a_i$  on the finite sets  $J$  of indices that are *order-connected* (if  $i < i' < i''$  and  $i, i'' \in J$  then  $i' \in J$ ). The next lemma records that this is the same notion.

**Lemma 3.2.** *Let  $a_0 < \dots < a_{k-1}$  be positive integers. The map  $J \mapsto \sum_{i \in J} a_i$  is injective on the order-connected subsets of  $\{0, \dots, k-1\}$  if and only if the sequence is valid.*

*Proof.* The order-connected subsets are the empty set and the blocks  $\{x, \dots, y-1\}$  with  $0 \leq x < y \leq k$ ; distinct pairs  $(x, y)$  give distinct sets. The empty set has sum 0, and every nonempty block has a positive sum, so the empty set never collides with a block. Hence injectivity on order-connected sets is injectivity of  $(x, y) \mapsto T(x, y)$  on  $0 \leq x < y \leq k$ .  $\square$

#### 4. THE UPPER BOUND

Throughout this section  $a_0 < a_1 < \dots < a_{k-1}$  is a valid sequence of integers in  $\{1, \dots, n\}$ , and  $A = \{a_0, \dots, a_{k-1}\}$ . Since the  $a_i$  are distinct positive integers,  $a_i \geq i+1$  and  $P(y) \geq y(y+1)/2$ .

##### 4.1. Canonical blocks and the crossing bound.

**Lemma 4.1** (Canonical blocks). *Let  $t$  be an integer with  $0 \leq 2t \leq n$ . For  $1 \leq y \leq k$  with  $P(y) > t$  let  $c(y)$  be the largest  $x < y$  with  $T(x, y) > t$ , and  $\sigma(y) = T(c(y), y)$ . Then  $t < \sigma(y) \leq n$ , and  $y \mapsto \sigma(y)$  is injective.*

*Proof.* Since  $T(0, y) = P(y) > t$ , the index  $c = c(y)$  exists, and  $\sigma(y) > t$  by definition. If  $c = y-1$  then  $\sigma(y) = a_{y-1} \leq n$ . Otherwise  $c+1 < y$ , and the maximality of  $c$  gives  $T(c+1, y) \leq t$ ; since the block  $[c+1, y)$  contains  $a_{y-1} > a_c$ , we get  $a_c < T(c+1, y) \leq t$ , so  $\sigma(y) = a_c + T(c+1, y) < 2t \leq n$ . If  $\sigma(y) = \sigma(y')$ , validity gives  $[c(y), y) = [c(y'), y')$ , so  $y = y'$ .  $\square$

**Corollary 4.2** (Crossing bound). *For every  $n \geq 0$ ,  $f(n) \leq \lfloor (n+1)/2 \rfloor + \lfloor \sqrt{n} \rfloor$ .*

*Proof.* Take  $t = \lfloor n/2 \rfloor$ . The  $y \in [1, k]$  with  $P(y) > t$  are mapped injectively into  $(t, n]$  by Lemma 4.1, so there are at most  $n - t = \lfloor (n+1)/2 \rfloor$  of them. If  $P(y) \leq t$  then  $y^2 < y(y+1) \leq 2P(y) \leq 2t \leq n$ , so  $y \leq \lfloor \sqrt{n} \rfloor$ ; hence at most  $\lfloor \sqrt{n} \rfloor$  indices  $y$  have  $P(y) \leq t$ .  $\square$

This is a version of the argument of [10]; we claim no credit for it. It is tight in its main term only for sets like  $A = (n/2, n] \cap \mathbb{Z}$ , where every block with sum at most  $n$  is a single term. Any further gain must therefore come from coincidences of block sums larger than  $n$ ; we use sums of order  $n^2$ , of runs of consecutive integers.

##### 4.2. Lengthening a block by one term.

**Lemma 4.3** (One step to the left). *Let  $\theta$  be an integer with  $3\theta \leq 2n+4$ . If  $1 \leq x, x+2 \leq y \leq k$  and  $T(x, y) \leq \theta$ , then  $T(x-1, y) \leq n$ .*

*Proof.* Put  $e = a_{x-1}$  and  $v = T(x, y)$ . The block  $[x, y)$  has at least two terms, and  $a_x \geq e+1$ ,  $a_{x+1} \geq e+2$ , so  $v \geq 2e+3$ . Suppose  $e+v > n$ . Then  $e \geq n+1-v$ , so  $v \geq 2(n+1-v)+3$ , that is  $3v \geq 2n+5$ . But  $3v \leq 3\theta \leq 2n+4$ , a contradiction.  $\square$

*Remark 4.4.* The condition  $3\theta \leq 2n+4$  cannot be weakened to  $\theta \leq \lfloor (2n+6)/3 \rfloor$ : for  $n = 3m$  with  $m \geq 4$  and the valid sequence  $m-1, m, m+2$  the block  $\{m, m+2\}$  has sum  $2m+2 = \lfloor (2n+6)/3 \rfloor$ , and adding  $m-1$  gives  $3m+1 > n$ . We take  $\theta = \lfloor (2n+4)/3 \rfloor$  below, and this is why the holes of Proposition 4.8 are counted only in  $(n/2, 2n/3]$ .

**Lemma 4.5** (Leaving  $(t, \theta]$ ). *Let  $3\theta \leq 2n+4$ , and let  $c+2 \leq y \leq k$  with  $T(c, y) \leq \theta < P(y)$ . Then there is  $x' < c$  with  $\theta < T(x', y) \leq n$ .*

*Proof.* Since the terms are positive,  $x \mapsto T(x, y)$  is decreasing. Let  $x^*$  be the least  $x$  with  $T(x, y) \leq \theta$ ; then  $x^* \leq c$ , and  $x^* \geq 1$  because  $T(0, y) = P(y) > \theta$ . By minimality  $T(x^*-1, y) > \theta$ , and Lemma 4.3 (with  $x^*+2 \leq c+2 \leq y$ ) gives  $T(x^*-1, y) \leq n$ . Take  $x' = x^*-1 < c$ .  $\square$

**4.3. The accounting.** For integers  $t \leq \theta$  let  $H = \{z \in \mathbb{Z} : t < z \leq \theta, z \notin A\}$  be the set of *holes* of  $A$  in  $(t, \theta]$ .

**Proposition 4.6** (Accounting). *Let  $t, \theta$  be integers with  $2t \leq n$ ,  $t \leq \theta \leq n$  and  $3\theta \leq 2n + 4$ . Then*

$$k + |H| \leq (n - t) + \lfloor \sqrt{2\theta} \rfloor.$$

*Proof.* Let  $\mathcal{Y} = \{y \in [1, k] : P(y) > t\}$ , and let  $c$  and  $\sigma$  be as in Lemma 4.1; the set  $\Sigma = \sigma(\mathcal{Y})$  lies in  $(t, n]$  and has  $|\mathcal{Y}|$  elements. Split  $H = H_a \cup H_b$  with  $H_a = H \cap \Sigma$  and  $H_b = H \setminus \Sigma$ . Let

$$\mathcal{Y}_1 = \{y \in \mathcal{Y} : \sigma(y) \leq \theta, \sigma(y) \notin A\}, \quad \mathcal{Y}_2 = \{y \in \mathcal{Y}_1 : P(y) > \theta\}, \quad \mathcal{Y}'_1 = \mathcal{Y}_1 \setminus \mathcal{Y}_2.$$

Every  $z \in H_a$  is  $\sigma(y)$  for exactly one  $y$ , and that  $y$  lies in  $\mathcal{Y}_1$ ; so  $|H_a| \leq |\mathcal{Y}_1|$ .

Let  $y \in \mathcal{Y}_2$ . The canonical block  $[c(y), y]$  is not a single term, since otherwise  $\sigma(y) = a_{y-1} \in A$ ; so  $c(y) + 2 \leq y$ . As  $T(c(y), y) = \sigma(y) \leq \theta < P(y)$ , Lemma 4.5 gives  $x'_y < c(y)$  with  $\Psi(y) := T(x'_y, y) \in (\theta, n]$ . By validity,  $\Psi$  is injective on  $\mathcal{Y}_2$  (equal sums give equal blocks, hence equal right ends). Moreover  $\Psi(y) \notin \Sigma$ : if  $\Psi(y) = \sigma(y_2)$ , validity gives  $[x'_y, y] = [c(y_2), y_2]$ , so  $y_2 = y$  and  $x'_y = c(y)$ , contradicting  $x'_y < c(y)$ .

The sets  $\Sigma$ ,  $H_b$  and  $\Psi(\mathcal{Y}_2)$  are pairwise disjoint subsets of  $(t, n]$ :  $H_b \subseteq (t, \theta] \subseteq (t, n]$  avoids  $\Sigma$  by definition and avoids  $\Psi(\mathcal{Y}_2) \subseteq (\theta, n]$ , and  $\Psi(\mathcal{Y}_2)$  avoids  $\Sigma$ . Hence

$$n - t \geq |\mathcal{Y}| + |H_b| + |\mathcal{Y}_2| = |\mathcal{Y}| + |H_b| + |\mathcal{Y}_1| - |\mathcal{Y}'_1| \geq |\mathcal{Y}| + |H| - |\mathcal{Y}'_1|.$$

Finally,  $[1, k] \setminus \mathcal{Y}$  and  $\mathcal{Y}'_1$  are disjoint subsets of  $\{y \in [1, k] : P(y) \leq \theta\}$ , and as in Corollary 4.2 this set has at most  $\lfloor \sqrt{2\theta} \rfloor$  elements. Therefore

$$k + |H| = |\mathcal{Y}| + |[1, k] \setminus \mathcal{Y}| + |H| \leq (n - t) + |\mathcal{Y}'_1| + |[1, k] \setminus \mathcal{Y}| \leq (n - t) + \lfloor \sqrt{2\theta} \rfloor. \quad \square$$

**4.4. Runs with equal sums.** For integers  $\ell \geq 1$  and  $w \geq 0$  let

$$Y_{\ell, w} = \{\ell w, \ell w + 1, \dots, \ell w + \ell\}, \quad X_{\ell, w} = \{(\ell + 1)w + 1, \dots, (\ell + 1)w + \ell\},$$

runs of  $\ell + 1$  and of  $\ell$  consecutive integers. Their sums are equal:

$$\sum Y_{\ell, w} = (\ell + 1)\ell w + \frac{\ell(\ell + 1)}{2} = \sum X_{\ell, w}.$$

We call the pair  $(\ell, w)$  a *pattern* and  $Y_{\ell, w} \cup X_{\ell, w}$  its *support*.

**Lemma 4.7.** (a) *If  $c, c + 1, \dots, c + r - 1$  all lie in  $A$ , they are consecutive terms  $a_i, \dots, a_{i+r-1}$  of the sequence.* (b) *For no pattern  $(\ell, w)$  are  $Y_{\ell, w}$  and  $X_{\ell, w}$  both contained in  $A$ .* (c) *If  $t < \ell w$  and  $(\ell + 1)w + \ell \leq \theta$ , the support of  $(\ell, w)$  lies in  $(t, \theta]$  and contains a hole.*

*Proof.* (a) If  $c$  and  $c + 1$  lie in  $A$ , no term lies strictly between them, so they are consecutive terms; induct on  $r$ . (b) Otherwise, by (a),  $Y_{\ell, w}$  and  $X_{\ell, w}$  are blocks of lengths  $\ell + 1$  and  $\ell$  with equal sums, contradicting validity. (c) The least element of the support is  $\ell w > t$  and the largest is  $(\ell + 1)w + \ell \leq \theta$ ; by (b) some element of the support is not in  $A$ .  $\square$

**4.5. A design with five bands.** Let  $j \geq 2$  be an integer and consider  $n' = 360000j$ , with

$$t = 180000j = n'/2, \quad \theta = 240000j + 1 = \lfloor (2n' + 4)/3 \rfloor.$$

A *band*  $(\ell; \alpha, \beta)$  is the set of patterns  $(\ell, w)$  with  $\alpha j < w \leq \beta j$ . We use the five bands

$$(4; 45000, 45687), \quad (5; 36550, 37499), \quad (6; 31250, 31729), \\ (5; 38075, 39999), \quad (9; 22222, 23999),$$

which contain  $(687 + 949 + 479 + 1924 + 1777)j = 5816j$  patterns in all (the two bands with  $\ell = 5$  have disjoint ranges of  $w$ ).

**Proposition 4.8.** *Let  $j \geq 2$ , and let  $a_0 < \dots < a_{k-1}$  be a valid sequence in  $\{1, \dots, 360000j\}$ . Then the set  $H$  of its holes in  $(180000j, 240000j + 1]$  has at least  $2908j$  elements.*

Only the case  $j \geq 12$  is used below, and only that case is formalized (`design_count`, Section 7); for  $2 \leq j \leq 11$  the proof that follows is a written proof only.

*Proof.* For a band  $(\ell; \alpha, \beta)$  the sets  $Y_{\ell,w}$  with  $\alpha j < w \leq \beta j$  lie in the  $Y$ -zone  $[\ell(\alpha j + 1), \ell\beta j + \ell]$  and the sets  $X_{\ell,w}$  in the  $X$ -zone  $[(\ell + 1)(\alpha j + 1) + 1, (\ell + 1)\beta j + \ell]$ . For the five bands these zones are:

| band              | Y-zone                       | X-zone                        |
|-------------------|------------------------------|-------------------------------|
| (4; 45000, 45687) | $[180000j + 4, 182748j + 4]$ | $[225000j + 6, 228435j + 4]$  |
| (5; 36550, 37499) | $[182750j + 5, 187495j + 5]$ | $[219300j + 7, 224994j + 5]$  |
| (6; 31250, 31729) | $[187500j + 6, 190374j + 6]$ | $[218750j + 8, 222103j + 6]$  |
| (5; 38075, 39999) | $[190375j + 5, 199995j + 5]$ | $[228450j + 7, 239994j + 5]$  |
| (9; 22222, 23999) | $[199998j + 9, 215991j + 9]$ | $[222220j + 11, 239990j + 9]$ |

For  $j \geq 2$  one checks directly:

- (1) every zone lies in  $(t, \theta] = (180000j, 240000j + 1]$ , so by Lemma 4.7(c) each of the  $5816j$  patterns has a hole in its support (the conditions  $t < \ell w$  and  $(\ell + 1)w + \ell \leq \theta$  hold at the extreme values of  $w$ , hence for all  $w$  in the band);
- (2) the five  $Y$ -zones are pairwise disjoint (in the order listed, each ends before the next begins; the third ends at  $190374j + 6$ , before  $190375j + 5$ , because  $j \geq 2$ ), and all of them lie below every  $X$ -zone ( $215991j + 9 < 218750j + 8$ );
- (3) no point lies in three  $X$ -zones. Write 5 and  $5'$  for the first and second band with  $\ell = 5$ . Ordered by left end, the  $X$ -zones are those of 6, 5, 9, 4,  $5'$ , and three inequalities separate them: the zone of 6 ends before that of 9 begins ( $222103j + 6 < 222220j + 11$ ), the zone of 5 ends before that of 4 begins ( $224994j + 5 < 225000j + 6$ ), and the zone of 4 ends before that of  $5'$  begins ( $228435j + 4 < 228450j + 7$ ). A zone that ends before another begins also ends before every zone that begins later, so the only pairs of  $X$ -zones that can meet are  $\{6, 5\}$ ,  $\{5, 9\}$ ,  $\{9, 4\}$  and  $\{9, 5'\}$ . These pairs contain no triangle, and three intervals on a line with a common point meet pairwise.

Now fix an integer  $z$  and count the patterns of the five bands whose support contains  $z$ . If  $z \in Y_{\ell,w}$  then  $\ell w \leq z \leq \ell w + \ell$ , so  $w \in \{ \lfloor z/\ell \rfloor - 1, \lfloor z/\ell \rfloor \}$ : at most two patterns of a band, and by (2) only one band's  $Y$ -zone and no  $X$ -zone contains  $z$ . If  $z \in X_{\ell,w}$  then  $(\ell + 1)w < z < (\ell + 1)(w + 1)$ , so  $w = \lfloor z/(\ell + 1) \rfloor$ : at most one pattern of a band, and by (3) at most two bands. So every  $z$  lies in the supports of at most two patterns.

By (1) choose for each pattern  $\omega$  a hole  $z_\omega \in H$  in its support. Then

$$5816j = \sum_{z \in H} \#\{\omega : z_\omega = z\} \leq \sum_{z \in H} \#\{\omega : z \in \text{supp } \omega\} \leq 2|H|. \quad \square$$

**4.6. Proof of Theorem 1.1.** Let  $a_0 < \dots < a_{k-1}$  be a valid sequence in  $\{1, \dots, n\}$  and  $r = \lfloor \sqrt{n} \rfloor$ . It suffices to show  $720000k \leq 354184n + 1440000r + 128010240000$ ; dividing by 720000 and using  $r \leq \sqrt{n}$  gives the second form, since  $354184/720000 = 44273/90000$  and  $128010240000/720000 = 177792$ .

*Case  $n < 4320000$ .* By Corollary 4.2,  $720000k \leq 360000(n + 1) + 720000r$ , and  $360000(n + 1) = 354184n + 5816n + 360000 < 354184n + 5816 \cdot 4320000 + 360000 < 354184n + 128010240000$ .

*Case  $n \geq 4320000$ .* Let  $j = \lceil n/360000 \rceil \geq 12$  and  $n' = 360000j$ , so  $n \leq n' \leq n + 359999$ . The sequence is valid in  $\{1, \dots, n'\}$ . With  $t = 180000j$  and  $\theta = 240000j + 1$  we have  $2t = n'$ ,

$t \leq \theta \leq n'$  and  $3\theta = 2n' + 3$ , so Propositions 4.6 and 4.8 give

$$k \leq (n' - t) - 2908j + \lfloor \sqrt{2\theta} \rfloor = 177092j + \lfloor \sqrt{480000j + 2} \rfloor.$$

Next,  $480000j + 2 = \frac{4}{3}n' + 2 \leq \frac{4}{3}(n + 359999) + 2 < \frac{4}{3}(r + 1)^2 + 480001 \leq (2r + 701)^2$ , the last step because  $(2r + 701)^2 - \frac{4}{3}(r + 1)^2 \geq 491401 - \frac{4}{3}$  for  $r \geq 0$ . Hence  $\lfloor \sqrt{480000j + 2} \rfloor \leq 2r + 700$ , and

$$\begin{aligned} 720000k &\leq 354184n' + 1440000r + 504000000 \\ &\leq 354184n + 1440000r + 354184 \cdot 359999 + 504000000. \end{aligned}$$

Since  $354184 \cdot 359999 + 504000000 = 128009885816 \leq 128010240000$ , the theorem follows.  $\square$

*Remark 4.9* (A simpler design). The two-band design with  $\ell \in \{5, 9\}$ , with ranges of  $w$  chosen as functions of  $n$  rather than of a multiple of 360000, gives  $1620f(n) \leq 797n + 1620\lfloor \sqrt{2[(2n + 4)/3]} \rfloor + 8000$  for all  $n$ , hence  $f(n) \leq \frac{797}{1620}n + 2\sqrt{n} + 7$ ; this is formally verified (`f_le_upper2` and `f_le_real2`, Section 7). We give only an outline, not a complete written proof. The proof is the one above with Proposition 4.8 replaced by a count of the patterns  $(5, w)$ ,  $\lfloor t/5 \rfloor < w \leq \lfloor (\theta - 5)/6 \rfloor$ , and  $(9, w)$ ,  $\lfloor 5\lfloor (\theta - 5)/6 \rfloor + 5 \rfloor/9 < w \leq \lfloor (\theta - 9)/10 \rfloor$ , where  $t = \lfloor n/2 \rfloor$  and  $\theta = \lfloor (2n + 4)/3 \rfloor$ ; again every integer lies in the supports of at most two patterns. The formal proof checks the eight floor inequalities this needs, and the closing arithmetic, by linear integer arithmetic for  $n \geq 400$  and by evaluation for  $12 \leq n < 400$ ; for  $n < 12$  the bound follows from Corollary 4.2. Nothing else in the paper depends on this remark except the comparison of the two bounds. Since  $\frac{797}{1620} - \frac{44273}{90000} = \frac{43}{810000}$  and both real forms have the term  $2\sqrt{n}$ , the bound of Theorem 1.1 is the smaller of the two exactly when  $\frac{43}{810000}n > 177792 - 7$ , that is for  $n \geq 3348973256$ .

*Remark 4.10* (What limits this argument). Two features of the proof cap what it can give. First, the holes are counted only in  $(t, \theta]$ , and Remark 4.4 shows that Lemma 4.3 fails beyond  $\theta$ ; a two-term canonical block with sum in  $(2n/3, 3n/4]$  can have a left neighbour that makes the sum exceed  $n$ . Second, the hole count uses only the run pairs  $(Y_{\ell, w}, X_{\ell, w})$  with supports in zones of bounded overlap. Numerical experiments recorded in the research repository are not part of the proofs and were not re-run for this paper. In them, the linear-programming optimum over interval designs was  $0.008105 \dots n$  holes, against  $\frac{2908}{360000}n = 0.00807 \dots n$  here. That model used run pairs of lengths  $p$  and  $q$  with  $q \leq 24$  and  $1 < p/q < 4/3$ , on a grid of 240 cells. The fractional hitting-set optimum over all equal-sum run pairs inside  $(t, \theta]$ , computed only at  $n = 3000$ , was about 30.3 holes, that is about  $0.0101n$ . These computations are exploratory, and the second, at a single small  $n$ , does not give an asymptotic constant. They suggest that the interval accounting used here is close to its limit, but we do not claim that the method is limited to any particular constant.

## 5. THE MODULUS-42 PATTERN

Let  $(r_0, \dots, r_5) = (2, 7, 11, 25, 26, 34)$  and  $g_t = 42\lfloor t/6 \rfloor + r_{t \bmod 6}$  for  $t \geq 0$ , so that

$$(g_t)_{t \geq 0} = 2, 7, 11, 25, 26, 34, 44, 49, 53, 67, 68, 76, 86, \dots, \quad g_{t+6} = g_t + 42.$$

The increments  $g_{t+1} - g_t$  are 5, 4, 14, 1, 8, 10 periodically, so  $g$  is strictly increasing. Write

$$(1) \quad r_c = 7c + \varepsilon_c, \quad (\varepsilon_0, \dots, \varepsilon_5) = (2, 0, -3, 4, -2, -1), \quad \sum_c \varepsilon_c = 0,$$

and extend  $\varepsilon$  6-periodically. Then  $g_t = 7t + \varepsilon_t$  and  $-3 \leq \varepsilon_t \leq 4$ . Put  $\pi(t) = \sum_{i < t} \varepsilon_i$ ;  $\pi$  is 6-periodic with  $(\pi(0), \dots, \pi(5)) = (0, 2, 2, -1, 3, 1)$ , so  $-1 \leq \pi \leq 3$  and  $\varepsilon_t = \pi(t + 1) - \pi(t)$ .

**Lemma 5.1** (Residue spectra). *Let  $w_0, w_1, \dots$  be integers with  $w_t \equiv g_t \pmod{42}$  for all  $t$ . If two blocks of  $(w_t)$  have the same sum, their lengths are congruent modulo 12.*

*Proof.* Let  $G(t) = \sum_{i < t} r_{i \bmod 6} \pmod{42}$ . Since  $r_0 + \dots + r_5 = 105 \equiv 21$ , we have  $G(t + 12) = G(t)$ , and

$$(G(0), \dots, G(11)) = (0, 2, 9, 20, 3, 29, 21, 23, 30, 41, 24, 8).$$

A block sum  $\sum_{t=x}^{x+\ell-1} w_t$  is congruent to  $G(x+\ell) - G(x)$ , which lies in the *spectrum*  $D_{\ell \bmod 12}$ , where  $D_\lambda = \{G(x+\lambda) - G(x) : 0 \leq x < 12\} \subseteq \mathbb{Z}/42\mathbb{Z}$ . Direct computation gives

|       |                              |          |                             |
|-------|------------------------------|----------|-----------------------------|
| $D_0$ | $\{0\}$                      | $D_6$    | $\{21\}$                    |
| $D_1$ | $\{2, 7, 11, 25, 26, 34\}$   | $D_7$    | $\{4, 5, 13, 23, 28, 32\}$  |
| $D_2$ | $\{9, 18, 36\}$              | $D_8$    | $\{15, 30, 39\}$            |
| $D_3$ | $\{1, 20\}$                  | $D_9$    | $\{22, 41\}$                |
| $D_4$ | $\{3, 12, 27\}$              | $D_{10}$ | $\{6, 24, 33\}$             |
| $D_5$ | $\{10, 14, 19, 29, 37, 38\}$ | $D_{11}$ | $\{8, 16, 17, 31, 35, 40\}$ |

These twelve sets are pairwise disjoint, as one reads off the table, and their sizes add up to  $1 + 6 + 3 + 2 + 3 + 6 + 1 + 6 + 3 + 2 + 3 + 6 = 42$ , so they partition  $\mathbb{Z}/42\mathbb{Z}$ . Hence the residue of a block sum determines its length modulo 12.  $\square$

*Proof of Theorem 1.2.* For  $m = 0$  there is nothing to prove; let  $m \geq 1$ ,  $k = 12m$ ,  $B = 21(m^2 + m + 2)$  (a multiple of 42, since  $m^2 + m$  is even) and  $a_t = B + g_t$  for  $0 \leq t < k$ . The sequence is strictly increasing,  $a_0 = B + 2 \geq 1$ , and  $a_{k-1} = B + 42(2m - 1) + 34 = 21m^2 + 105m + 34$ .

Suppose it is not valid, and take  $x, \ell, y, s$  as in Lemma 3.1. By Lemma 5.1 (with  $w_t = a_t$ , since  $42 \mid B$ ) we have  $\ell \equiv s \pmod{12}$ , so  $\ell \geq s + 12$ . Since  $g$  is increasing and nonnegative,

$$S(x, \ell) = \ell B + \sum_{t=x}^{x+\ell-1} g_t \geq \ell B + \sum_{t < \ell} g_t \geq (s + 12)B + \sum_{t < s} g_t.$$

On the other hand  $g_{y+t} - g_t = 7y + \varepsilon_{y+t} - \varepsilon_t \leq 7y + 7$ , so

$$S(y, s) = sB + \sum_{t < s} g_{y+t} \leq sB + \sum_{t < s} g_t + s(7y + 7).$$

Equality of the two sums therefore gives  $12B \leq s(7y + 7) \leq 7s(12m + 1 - s)$ , using  $y \leq k - s$ . The right side is at most  $\frac{7}{4}(12m + 1)^2 = 252m^2 + 42m + \frac{7}{4}$ , while  $12B = 252m^2 + 252m + 504$ . This contradiction shows the sequence is valid, so  $f(21m^2 + 105m + 46) \geq f(21m^2 + 105m + 34) \geq 12m$ .

For the bound in  $n$ , let  $Q = \lfloor \sqrt{n/21} \rfloor$ . Then  $n < 21(Q + 1)^2$ , so  $\frac{12}{\sqrt{21}}\sqrt{n} < 12(Q + 1)$ . If  $Q \leq 2$  the claimed bound is negative. If  $Q \geq 3$ , put  $m = Q - 3$ ; then  $21m^2 + 105m + 46 \leq 21(m + 3)^2 = 21Q^2 \leq n$ , so  $f(n) \geq 12m = 12(Q + 1) - 48 > \frac{12}{\sqrt{21}}\sqrt{n} - 48$ .  $\square$

*Remark 5.2.* Consider a periodic pattern: an offset plus  $q\lfloor t/p \rfloor + r_{t \bmod p}$  for residues  $r_0 < \dots < r_{p-1}$  modulo  $q$ , and let the *gap*  $\gamma$  be the least  $\gamma \geq 1$  such that two blocks of lengths differing by  $\gamma$  can have congruent sums modulo  $q$  (for the pattern above,  $\gamma = 12$  by Lemma 5.1). Heuristically, following the plan of [19], an offset of about  $\frac{q}{4p\gamma}k^2$  suffices for the size argument when the inseparable length differences are the multiples of  $\gamma$ , and it yields the constant  $2\sqrt{p\gamma/q}$ . We call  $p\gamma/q$  the *score* of the pattern. Here  $p = 6$ ,  $q = 42$ ,  $\gamma = 12$  give score  $12/7$  and constant  $2\sqrt{12/7} = 12/\sqrt{21}$ ; Pickhardt's pattern ( $q = 3$ ,  $p = 2$ ,  $\gamma = 2$ ) gives  $4/\sqrt{3}$ , and the modulus-22 pattern  $(2, 6, 9, 12, 18, 19)$  ( $p = \gamma = 6$ , score  $18/11$ ) gives  $12/\sqrt{22}$ . For the modulus-42 pattern the offset of Theorem 1.2 is close to necessary: numerically, offsets  $42t$  still produce coincidences up to 0.84, 0.94 and 0.96 times  $21(m^2 + m + 2)$  for  $m = 20, 60$  and  $100$ .

## 6. THE SKIP CONSTRUCTION

Fix an integer  $j \geq 1$  and put

$$k = 360j, \quad b = 288j = \frac{4}{5}k.$$

For an integer  $u \geq 0$  consider

$$(2) \quad a_t = 42u + g_t + 84 \cdot [t \geq b] \quad (0 \leq t < k),$$

where  $[t \geq b]$  is 1 if  $t \geq b$  and 0 otherwise. The sequence is strictly increasing and positive; adding  $84 = 2 \cdot 42$  from index  $b$  on has the same effect as deleting twelve consecutive terms (two periods) of the pure pattern. Write  $\nu(z) = \max(z - b, 0)$  for the number of indices  $t < z$  with  $t \geq b$ .

**Lemma 6.1** (Block sums). *For  $x \geq 0$ ,  $\ell \geq 0$  with  $x + \ell \leq k$ ,*

$$2S(x, \ell) = 84u\ell + 7\ell(2x + \ell - 1) + 2(\pi(x + \ell) - \pi(x)) + 168(\nu(x + \ell) - \nu(x)).$$

*Proof.* By (1),  $a_t = 42u + 7t + (\pi(t + 1) - \pi(t)) + 84[t \geq b]$ . Sum over  $x \leq t < x + \ell$ : the second term gives  $\frac{7}{2}\ell(2x + \ell - 1)$ , the third telescopes, and the fourth gives  $84(\nu(x + \ell) - \nu(x))$ .  $\square$

**Lemma 6.2** (Phase filter). *Let  $\xi, \eta, \lambda \in \mathbb{Z}/6\mathbb{Z}$  and  $\Delta = (\pi(\eta + \lambda) - \pi(\eta)) - (\pi(\xi + \lambda) - \pi(\xi))$ . If  $\Delta \equiv 0 \pmod{7}$  and  $(\eta - \xi)\lambda + \Delta \equiv 0 \pmod{6}$ , then  $\Delta = 0$ .*

*Proof.* Since  $\pi$  is 6-periodic, it is a function on  $\mathbb{Z}/6\mathbb{Z}$ , so  $\Delta$  is well defined. If  $\lambda = 0$ , both differences vanish. In general each difference  $\pi(\alpha + \lambda) - \pi(\alpha)$  lies in  $[-4, 4]$ , so  $|\Delta| \leq 8$  and, if  $\Delta \neq 0$ ,  $\Delta = \pm 7$ ; then one difference is  $\pm 4$  and the other  $\mp 3$ . Since  $\pi$  takes the value  $-1$  only at 3 and the value 3 only at 4, a difference  $+4$  occurs only for  $(\alpha, \lambda) = (3, 1)$  and a difference  $-4$  only for  $(\alpha, \lambda) = (4, 5)$ . For  $\lambda = 1$  the differences are  $\varepsilon_\alpha$ , which equals  $-3$  only at  $\alpha = 2$ ; for  $\lambda = 5$  they are  $-\varepsilon_{\alpha-1}$ , which equals 3 only at  $\alpha = 3$ . This leaves four cases:

| $\Delta$ | $(\xi, \eta, \lambda)$ | $(\eta - \xi)\lambda \pmod{6}$ | required, $\equiv -\Delta \pmod{6}$ |
|----------|------------------------|--------------------------------|-------------------------------------|
| +7       | (2, 3, 1)              | 1                              | 5                                   |
| +7       | (4, 3, 5)              | 1                              | 5                                   |
| -7       | (3, 4, 5)              | 5                              | 1                                   |
| -7       | (3, 2, 1)              | 5                              | 1                                   |

(the hypothesis requires  $(\eta - \xi)\lambda \equiv -\Delta \pmod{6}$ ). None is possible.  $\square$

**Proposition 6.3** (Reduction to a divisor equation). *Let  $j \geq 1$  and  $u \geq 288j^2 + 12j + 1$ . If the sequence (2) is not valid, then there are integers  $L, \delta$  with*

$$14 \leq L \leq \delta \leq b - 1 \quad \text{and} \quad L\delta = 72u + 66 + 12b.$$

*Proof.* Take  $x, \ell, y, s$  as in Lemma 3.1. All of  $42u$ ,  $42[t/6]$  and 84 vanish modulo 42, so Lemma 5.1 applies and  $h = \ell - s$  is a positive multiple of 12. Put  $d = y - x \geq \ell$ . Let

$$\Delta = (\pi(y + s) - \pi(y)) - (\pi(x + \ell) - \pi(x)) \in [-8, 8], \quad c_\ell = \nu(x + \ell) - \nu(x), \quad c_s = \nu(y + s) - \nu(y),$$

so  $c_\ell$  and  $c_s$  count the indices  $\geq b$  in the two blocks. Subtracting the identities of Lemma 6.1 for the two blocks and substituting  $\ell = s + h$ ,  $y = x + d$  gives

$$(3) \quad 84uh = 7(2sd - 2hx - 2hs - h^2 + h) + 2\Delta + 168(c_s - c_\ell).$$

As  $h \geq 1$  and  $x, s \geq 0$ , this implies

$$(4) \quad 84uh \leq 14sd + 16 + 168(c_s - c_\ell).$$

We use  $1008u \geq 1008(288j^2 + 12j + 1) = 290304j^2 + 12096j + 1008$  and distinguish four cases. They exhaust all possibilities: if  $x \geq b$  we are in (ii); otherwise  $x < b$ , and either  $y \geq b$  (case (iii)) or  $y < b$ , and then  $y + s \leq b$  (case (i)) or  $y + s > b$  (case (iv)).

(i)  $y + s \leq b$ . Then  $c_s = c_\ell = 0$  and  $s + d = y + s - x \leq b$ , so  $sd \leq b^2/4 = 20736j^2$ , and (4) with  $h \geq 12$  gives  $1008u \leq 290304j^2 + 16$ , a contradiction.

(ii)  $x \geq b$ . Then  $c_\ell = \ell$ ,  $c_s = s$ , so  $c_s - c_\ell < 0$ , and  $s + d \leq k - b = 72j$ ; (4) gives  $1008u \leq 14 \cdot (36j)^2 + 16$ , a contradiction.

(iii)  $x < b \leq y$ . Then  $c_s = s$ ,  $c_s - c_\ell \leq s$ ,  $s \leq k - y \leq 72j$  and  $d + s \leq k$ . By (4),  $1008u \leq 14s(k - s) + 168s + 16 = 14s(360j + 12 - s) + 16$ . The function  $s \mapsto s(360j + 12 - s)$  is increasing for  $s \leq 180j$ , so this is at most  $14 \cdot 72j(288j + 12) + 16 = 290304j^2 + 12096j + 16$ , a contradiction.

(iv)  $y < b < y + s$ . Then  $c_\ell = 0$  (the long block ends before  $y < b$ ) and  $c_s = y + s - b \leq s - 1$ ; in particular  $s \geq 2$ . If  $h \geq 24$ , (4) and  $s + d \leq k$  give  $2016u \leq 14(180j)^2 + 168 \cdot 360j + 16 = 453600j^2 + 60480j + 16$ , while  $2016u \geq 580608j^2 + 24192j + 2016$ ; a contradiction. Hence  $h = 12$ , and (3) with  $c_s - c_\ell = x + d + s - b$  simplifies to

$$(5) \quad 504u + 462 + 84b = 7d(s + 12) + \Delta.$$

Since  $7 \mid 504, 462, 84$  we get  $\Delta \equiv 0 \pmod{7}$ ; since  $6 \mid 504, 462, 84$  we get  $7d(s + 12) + \Delta \equiv ds + \Delta \equiv 0 \pmod{6}$ . Here  $\pi(x + \ell) = \pi(x + s)$  because  $\ell = s + 12$ , so  $\Delta$  depends only on  $x, y, s$  modulo 6, and  $d \equiv y - x$ . By Lemma 6.2 (with  $\xi \equiv x, \eta \equiv y$ , and the same  $\lambda \equiv s$  in both differences),  $\Delta = 0$ , and (5) becomes  $(s + 12)d = 72u + 66 + 12b$ . Take  $L = s + 12$  and  $\delta = d$ : then  $L \geq 14$ ,  $\delta = d \geq \ell = L$ , and  $\delta = y - x \leq y \leq b - 1$ .  $\square$

**Proposition 6.4** (Counting). *Let  $j \geq 230$  and  $U = [288j^2 + 12j + 1, 288j^2 + 2012j]$ , an interval of  $2000j$  integers. Some  $u \in U$  admits no pair  $(L, \delta)$  as in Proposition 6.3.*

Only the case  $j \geq 500$  is used below, and only that case is formalized (`exists_good`); see Remark 6.5.

*Proof.* Call  $u \in U$  bad if a pair exists, and put  $M(u) = 72u + 66 + 12b = 72u + 66 + 3456j$ . For  $u \in U$ ,  $P_1 \leq M(u) \leq P_2$  with  $P_1 = 20736j^2 + 4320j + 138$  and  $P_2 = 20736j^2 + 148320j + 66$ ; note  $P_2 - P_1 = 72(2000j - 1)$ . Since  $u \mapsto M(u)$  is injective, the number of bad  $u$  is at most the number of pairs  $(L, \delta)$  with  $14 \leq L \leq \delta \leq 288j - 1$  and  $P_1 \leq L\delta \leq P_2$ ,  $L\delta \equiv 66 \pmod{72}$ .

*Range of  $L$ .* If  $L \leq 72j - 1$  then  $L\delta < (L + 1)(\delta + 1) \leq 72j \cdot 288j = 20736j^2 < P_1$ . If  $L \geq 144j + 515$  then  $L\delta \geq L^2 \geq (144j + 515)^2 > P_2$ . So  $72j \leq L \leq 144j + 514$ , i.e.  $L = 72\mu + \rho$  with  $0 \leq \rho < 72$  and  $j \leq \mu \leq 2j + 7$ .

*Count for fixed  $L$ .* Let  $\kappa(\rho)$  be the number of residues  $r$  modulo 72 with  $\rho r \equiv 66 \pmod{72}$ . For fixed  $L$  the cofactor  $\delta$  lies in one of  $\kappa(L \bmod 72)$  residue classes modulo 72 and in the interval  $[P_1/L, P_2/L]$ , of length  $72(2000j - 1)/L$ , so each class contributes at most  $\lfloor (2000j - 1)/L \rfloor + 1$  values. Moreover  $\sum_{\rho < 72} \kappa(\rho) = \#\{(\rho, r) \in (\mathbb{Z}/72)^2 : \rho r = 66\} = 8 \cdot 12 = 96$ , by the Chinese remainder theorem:  $\rho r \equiv 2 \pmod{8}$  has 8 solutions and  $\rho r \equiv 3 \pmod{9}$  has 12.

*Summation.* Using  $L \geq 72\mu$ ,

$$\#\{\text{bad } u\} \leq \sum_{\mu=j}^{2j+7} \sum_{\rho < 72} \kappa(\rho) \left( \frac{2000j}{72\mu} + 1 \right) = \frac{8000}{3} j \sum_{\mu=j}^{2j+7} \frac{1}{\mu} + 96(j + 8).$$

By  $\frac{1}{\mu} \leq \log \mu - \log(\mu - 1)$  for  $\mu > j$  and  $\log(1 + x) \leq x$ ,  $\sum_{\mu=j}^{2j+7} \frac{1}{\mu} \leq \frac{1}{j} + \log \frac{2j+7}{j} \leq \log 2 + \frac{9}{2j}$ . Hence, with  $\log 2 < 0.6931471808$ ,

$$\#\{\text{bad } u\} \leq 12000 + \frac{8000}{3}(\log 2)j + 96j + 768 < 1944.4j + 12768 < 2000j$$

for  $j \geq 230$ .  $\square$

*Proof of Theorem 1.3.* Let  $j \geq 500$  and choose  $u \in U$  as in Proposition 6.4. By Proposition 6.3 the sequence (2) is valid. It has  $k = 360j$  terms, the smallest is  $a_0 = 42u + 2 \geq 1$ , and the largest is

$$a_{k-1} = 42u + 42(60j - 1) + 34 + 84 \leq 42(288j^2 + 2012j) + 2520j + 76 = 12096j^2 + 87024j + 76.$$

Hence  $f(12096j^2 + 87024j + 81) \geq 360j$ .

For  $n \geq 12096 \cdot 504^2$  let  $Q = \lfloor \sqrt{n/12096} \rfloor \geq 504$  and  $j = Q - 4 \geq 500$ . Then  $12096j^2 + 87024j + 81 \leq 12096(j + 4)^2 = 12096Q^2 \leq n$ , so  $f(n) \geq 360j$ . Also  $n < 12096(Q + 1)^2$  and  $12096 = 21 \cdot 24^2$ , so  $\frac{15}{\sqrt{21}}\sqrt{n} < \frac{15}{\sqrt{21}} \cdot 24\sqrt{21}(Q + 1) = 360(Q + 1) = 360j + 1800$ . For  $n < 12096 \cdot 504^2$ ,  $\frac{15}{\sqrt{21}}\sqrt{n} < 360 \cdot 504 = 181440$ , so the all- $n$  form is trivial there.  $\square$

*Remark 6.5.* The proof of Proposition 6.4 works for  $j \geq 230$ ; the formal proof uses  $j \geq 500$ . Proposition 6.3 holds for every  $j \geq 1$ , and a finite computation (Section 8; not formalized) shows that for every  $j \leq 499$  the interval  $U$  also contains an offset with no pair  $(L, \delta)$ . With it,  $f(12096j^2 + 87024j + 81) \geq 360j$  for every  $j \geq 1$ , and the argument above would give  $f(n) \geq \frac{15}{\sqrt{21}}\sqrt{n} - 1800$  for *all*  $n$ . We do not use this, because the finite computation is not part of the verified development; the threshold  $n \geq 12096 \cdot 504^2$  and the constant 181440 in Theorem 1.3 are artifacts of the formal proof.

*Remark 6.6* (Where  $15/\sqrt{21}$  comes from). With  $k$  and  $b$  free, the four cases of Proposition 6.3 need, to leading order,  $u > b^2/288$  (case (i)),  $u > (k - b)^2/288$  (case (ii)),  $u > b(k - b)/72$  (case (iii), for  $b \geq k/2$ ) and  $u > k^2/576$  (case (iv) with  $h \geq 24$ ). The largest of these is smallest at  $b = \frac{4}{5}k$ , where (i) and (iii) both give  $u > k^2/450 = 288j^2$ . Then the largest term is about  $42u \approx \frac{7}{75}k^2$ , and  $k/\sqrt{7k^2/75} = \sqrt{75/7} = 15/\sqrt{21}$ . The counting step costs only a window of  $O(j)$  offsets and so affects only lower-order terms. This is an optimization of our sufficient conditions, not a proof that the family cannot do better; the mirror placement  $b < k/2$  was not examined. In exact computations for  $j = 2$  and  $j = 3$ , coincidences not of the divisor type of case (iv) with  $h = 12$  occur for offsets up to  $u = 1131$  and  $u = 2561$ , within 4 per cent of the left ends 1177 and 2629 of  $U$ . But some offsets below the left end already give valid sequences (the smallest valid offsets are  $u = 1069$  for  $j = 2$  and  $u = 2457$  for  $j = 3$ , about 0.91 and 0.93 times the left end), so the thresholds are not sharp, and we do not know the limit of the family.

## 7. FORMAL VERIFICATION

**7.1. What was verified.** The formal development consists of six Lean 4 [7] files, which import only Mathlib [23]:

| file                | content                                                                       |
|---------------------|-------------------------------------------------------------------------------|
| P0357.lean          | definitions; Lemmas 3.1, 5.1; Theorem 1.2; Corollary 4.2                      |
| P0357Skip.lean      | Section 6; Theorem 1.3                                                        |
| P0357Upper.lean     | Lemmas 3.2–4.5; Proposition 4.6                                               |
| P0357Upper2.lean    | Lemma 4.7; Remark 4.9                                                         |
| P0357Upper4Gen.lean | double counting over bands (the last display of the proof of Proposition 4.8) |
| P0357Upper4.lean    | the five bands, their degree bound, Proposition 4.8, Theorem 1.1              |

Besides what the table lists, P0357.lean contains further results on Problem 357 that this paper does not use: the modulus-22 pattern, a reformulation of validity through additive energy, density statements for infinite sequences, and examples of arithmetic progressions. The axiom audit covers them too. The definitions are (with mathematical symbols for the Unicode of the source)

```
def HasDistinctSums { $\iota$   $\alpha$  : Type*} [Preorder  $\iota$ ] [AddCommMonoid  $\alpha$ ]
  (a :  $\iota \rightarrow \alpha$ ) : Prop :=
  {J : Finset  $\iota$  | (J : Set  $\iota$ ).OrdConnected}.InjOn (fun J  $\mapsto \sum x \in J, a x$ )
```

```
noncomputable def f (n :  $\mathbb{N}$ ) :  $\mathbb{N}$  :=
  sSup {k :  $\mathbb{N}$  |  $\exists a : \text{Fin } k \rightarrow \mathbb{Z},$ 
    Set.range a  $\subseteq$  Set.Icc 1 n  $\wedge$  StrictMono a  $\wedge$  HasDistinctSums a}
```

The verified statements include (namespaces omitted;  $\sqrt{\phantom{x}}$  is the real square root and `n.sqrt` the integer one)

```
theorem f_le_upper4 (n :  $\mathbb{N}$ ) :
  720000 * f n  $\leq$  354184 * n + 1440000 * n.sqrt + 128010240000
theorem f_le_real4 (n :  $\mathbb{N}$ ) : (f n :  $\mathbb{R}$ )  $\leq$  44273 / 90000 * n + 2 *  $\sqrt{n}$  + 177792
theorem f_le_real2 (n :  $\mathbb{N}$ ) : (f n :  $\mathbb{R}$ )  $\leq$  797 / 1620 * n + 2 *  $\sqrt{n}$  + 7
theorem mod42_lower (m :  $\mathbb{N}$ ) : 12 * m  $\leq$  f (21 * m2 + 105 * m + 46)
theorem mod42_lower_real (n :  $\mathbb{N}$ ) : 12 /  $\sqrt{21}$  *  $\sqrt{n}$  - 48  $\leq$  (f n :  $\mathbb{R}$ )
theorem skip_lower (j :  $\mathbb{N}$ ) (hj : 500  $\leq$  j) :
  360 * j  $\leq$  f (12096 * j2 + 87024 * j + 81)
theorem skip_lower_real_large (n :  $\mathbb{N}$ ) (hn : 12096 * 5042  $\leq$  n) :
  15 /  $\sqrt{21}$  *  $\sqrt{n}$  - 1800  $\leq$  (f n :  $\mathbb{R}$ )
theorem skip_lower_real (n :  $\mathbb{N}$ ) : 15 /  $\sqrt{21}$  *  $\sqrt{n}$  - 181440  $\leq$  (f n :  $\mathbb{R}$ )
```

and the intermediate results `main_count` (Proposition 4.6, for arbitrary  $t, \theta$  with the stated hypotheses), `design_count` (Proposition 4.8, stated there for  $j \geq 12$ ), `collision_pair` and `distinct_skip` (Proposition 6.3), `exists_good` (Proposition 6.4, for  $j \geq 500$ ) and `f_le_half_add_sqrt` (Corollary 4.2). The file `verification/Headline.lean` of the repository [21] prints every statement used here.

**7.2. Interpretation: does the formal theorem state the problem?** A proof assistant checks that the proofs prove the stated theorems; it does not check that the statements mean what we intend. We therefore record the correspondence explicitly.

- (1) *Same definition as the community statement.* Our `HasDistinctSums` and `f` are definitionally equal to the definitions in `FormalConjectures/ErdosProblems/357.lean` of the Formal Conjectures project [14], at the commit `d16e05a` (12 August 2026) used for the development and at the later commit `4b5e527` (15 October 2026). This is checked mechanically: a script copies the two upstream definitions verbatim into a separate namespace, and Lean accepts the equations `Erdos357.f = Upstream.f` and `Erdos357.HasDistinctSums a  $\leftrightarrow$  Upstream.HasDistinctSums a` by `rfl`.
- (2) *Same notion as in the introduction.* By Lemma 3.2, `HasDistinctSums` for a strictly increasing positive sequence is validity. The formal sequences are maps `Fin k  $\rightarrow$   $\mathbb{Z}$`  with values in  $[1, n]$  and `StrictMono`, that is,  $1 \leq a_0 < \dots < a_{k-1} \leq n$ .
- (3) *The supremum is a maximum.* The supremum is taken in  $\mathbb{N}$ , where an unbounded set has supremum 0. The set in question contains 0 and is bounded by  $n$ , so its supremum is its maximum; the formal proofs of the lower bounds establish boundedness before using the supremum, and the upper bounds are proved for every element of the set, so neither direction relies on the convention.
- (4) *No special case.* The theorems are stated for the function  $f$  itself, for all  $n$  (Theorems 1.1 and 1.2) and for all  $n \geq 12096 \cdot 504^2$  or all  $n$  with the weaker constant (Theorem 1.3); they are not stated for a restricted class of sequences. They do not address the open question, Formal Conjectures' `erdos_357.parts.i` ( $f(n) = o(n)$ ), which remains unproved.

**7.3. How it was checked.** We rebuilt the six files in a fresh Lean project with Lean v4.27.0 and Mathlib at commit a3a10db (tag v4.27.0); the build log, the exact toolchain and every check below are in the repository [21], together with a script that repeats them.

- *Axioms.* For every declaration of the six files, not only the theorems listed above, we computed the axioms on which it depends. All depend only on the standard axioms `propext`, `Classical.choice` and `Quot.sound`; none uses `sorryAx` (an unfinished proof) or `Lean.ofReduceBool` (`native_decide`, which trusts compiled code), and the files declare no axioms.
- *Kernel replay.* Every declaration was re-checked by the Lean kernel from the compiled files with `lean4checker` [17] (tag v4.27.0). This re-checks the stored proof terms without trusting the elaborator, the tactics or the build cache. It uses the same kernel implementation as the build, so it is a check of the build, not an independent kernel.
- *Finite case checks inside the proofs.* The spectra of Lemma 5.1 are checked by a kernel-evaluated decision over  $12^4$  cases, Lemma 6.2 over its 216 cases, and the residue count 96 of Proposition 6.4 by kernel evaluation; the harmonic estimate uses  $\log x \leq x - 1$  and Mathlib’s bound  $\log 2 < 0.6931471808$ .
- *Generated code.* The file `P0357Upper4.lean` (the five bands and the case split behind the degree bound of Proposition 4.8, 19 breakpoints) was generated by a Python script from a data file listing the bands. The script is not trusted: Lean checks its output like any other file, and a wrong design would fail to compile. The script and data file are in the repository.

The development was written for the research repository in which the results were found, and the files are copied from it unchanged. That repository is not public; comments in the Lean files that name its research notes are not needed to read or check the proofs. Three further files of that repository about Problem 357 are not needed here and are omitted: one on side questions, one proving a conditional reduction, and one recording a negative result on extending Lemma 4.3.

## 8. FINITE COMPUTATIONS

The written proofs of Theorems 1.1–1.3 are complete; the computations in this section are not used in them. They serve as consistency checks and as data for small  $n$ . Each item says how it was checked; “exhaustive search” results are computational evidence, not formal proofs.

### Checks of the proofs.

- (1) Every numerical claim and finite case analysis in Sections 4–6 was re-checked with exact integer and rational arithmetic in a script independent of the Lean development: the zones and degree bound of Proposition 4.8 (including a direct count of the patterns through every integer for  $2 \leq j \leq 6$ ), the arithmetic of the proof of Theorem 1.1, the spectra and the 216 cases of Lemmas 5.1 and 6.2, the identity of Lemma 6.1, the constants of the four cases of Proposition 6.3, the count 96, and the union bound of Proposition 6.4. An adversarial review recomputed them again with separately written code, and tested the lemmas of Sections 3 and 4 on every valid sequence in  $\{1, \dots, 22\}$ .
- (2) Theorem 1.2: all block sums of the sequence are distinct for  $1 \leq m \leq 60$ .
- (3) Proposition 6.3, for  $j = 1, 2, 3, 4$  and every  $u \in U$ : all block sums were computed. Every  $u$  with no pair  $(L, \delta)$  gives a valid sequence, and every coincidence found for the other values of  $u$  is of the type in case (iv):  $h = 12$ , the short block contains  $b$ , and  $(s + 12)d = 72u + 66 + 12b$ .

- (4) For every  $j \leq 499$  the interval  $U$  contains an offset with no pair  $(L, \delta)$  (the smallest proportion of such offsets is 0.638, at  $j = 16$ ). This is the finite check mentioned in Remark 6.5.
- (5) The exact number of bad  $u$  is 337,866 of  $10^6$  for  $j = 500$ , against the bound  $1944.4j + 12768$  of Proposition 6.4; the counting step has about a factor 3 to spare.

**Exact values for small  $n$ .** Let  $M(k)$  be the least  $n$  with  $f(n) \geq k$ . Exhaustive computer search gives

$$\begin{aligned} (M(1), \dots, M(14)) &= (1, 2, 4, 5, 7, 10, 12, 13, 15, 18, 21, 24, 25, 29), \\ (M(15), \dots, M(28)) &= (30, 33, 36, 38, 41, 47, 50, 52, 56, 60, 63, 67, 69, 75), \end{aligned}$$

and  $f(76) = f(77) = 28$ ; together these determine  $f(n)$  for every  $n \leq 77$ . The values for  $k \leq 22$  agree with [20] and those for  $n \leq 53$  with [5]. Each value  $M(k)$  consists of a lower bound, a valid  $k$ -term sequence with largest term  $M(k)$ , which anyone can check from the definition, and an upper bound, the nonexistence of such a sequence with largest term  $M(k) - 1$ , which rests on the correctness of the search. The values were first obtained by two exhaustive programs with different pruning rules, which agree on every value; for  $k \leq 27$  their nonexistence claims were also repeated with all pruning disabled, but the remaining ones ( $f(71) = \dots = f(74) = 27$  and  $f(76) = f(77) = 28$ ) used a pruning rule based on previously computed values  $M(r)$ . For this paper a third program was written from the definition alone; the earlier programs were not consulted while it was written, and their results were used only for the final comparison. Its only pruning rules are that each term must leave room for the later ones and that each later term must differ from every block sum of the terms before it, and it uses no previously computed value. It reproduces  $M(k)$  for every  $k \leq 28$ , the lists of all sequences attaining them (67 in all), and  $f(n)$  for every  $n \leq 77$ , including the nonexistence claims above. Its pruned search agrees with a complete unpruned enumeration of all valid sequences in  $\{1, \dots, 52\}$  and with a brute force over all subsets of  $\{1, \dots, 34\}$ . All of this is computational evidence. For example,

$$2, 3, 4, 6, 11, 25, 38, 40, 41, 47, 52, 54, 56, 57, 58, 60, 61, 62, 64, 66, 67, 68, 69, 70, 71, 72, 73, 75$$

is one of the two valid 28-term sequences with largest term 75.

**Greedy sequences.** Greedy constructions produce valid sequences much longer than our lower bounds for every  $n$  we could reach. A reverse greedy construction gives a valid sequence of  $26813 = 5.99 \dots \sqrt{n}$  terms in  $\{1, \dots, n\}$ ,  $n = 2 \cdot 10^7$ , against 11676 terms from Theorem 1.2's construction at the same  $n$ . The sequence was regenerated deterministically for this paper, and two separately written programs confirmed that it is strictly increasing and that all  $26813 \cdot 26814/2 = 359,481,891$  of its block sums are distinct. We do not know whether  $f(n)/\sqrt{n}$  is bounded, and finite data do not decide it.

## 9. HOW THE RESULTS WERE FOUND

The work was done by AI agents, as described in the statement on AI use. We summarize what the records show, because it explains the shape of the constructions.

*Lower bounds.* Pickhardt's construction is a periodic pattern in the sense of Remark 5.2, with  $q = 3$  and  $p = 2$ . On 9 September 2026 an exhaustive search over all residue sets modulo  $q \leq 22$ , ranked by the score of Remark 5.2, found the modulus-22 pattern  $(2, 6, 9, 12, 18, 19)$  and a formal proof of  $f(n) \geq \frac{12}{\sqrt{22}}\sqrt{n} - 36$ . On 23 October 2026 a branch-and-bound search for  $23 \leq q \leq 60$  found patterns of score above 18/11 only for  $q = 42$ : four residue sets with  $p = 6$  and  $\gamma = 12$ , namely  $R, -R, R+21$  and  $-R+21$  modulo 42, where  $R = \{2, 7, 11, 25, 26, 34\}$ ; the

search record also reports no pattern of score above  $12/7$  for  $61 \leq q \leq 78$ . These search results are not needed for the proofs. The skip arose from a numerical experiment: adding a multiple of  $q$  to all terms beyond some index keeps every residue, and the smallest admissible offset was smallest with the jump near  $0.8k$ . The written proof of Section 6 and its formalization came afterwards; an intermediate version used a window of  $4j^2$  offsets and gave the constant  $360/\sqrt{12264} = 3.2507\dots$

*Upper bound.* The crossing argument (Corollary 4.2) was taken from [10]. The left-extension lemma and the accounting of Proposition 4.6 were found first with a single family of run pairs,  $\ell = 6$ , giving  $f(n) \leq \frac{125}{252}n + 2\sqrt{n} + 4$ ; replacing it by two families of degree two gave  $\frac{797}{1620}$  (Remark 4.9). A linear program over band designs, rounded to the grid  $360000j$ , gave the five bands of Proposition 4.8.

## 10. LIMITATIONS

- The main question, whether  $f(n) = o(n)$ , is untouched. Our upper bound improves the leading constant from  $\frac{1}{2}$  to  $\frac{44273}{90000}$ , a gain of  $\frac{727}{90000} = 0.00807\dots$ ; Remark 4.10 explains why we expect this accounting to stop near there.
- The upper bound of Theorem 1.1 is below  $n/2$  only for  $n \geq 23202647$ , and its additive constant 177792 is an artifact of rounding the design to multiples of 360000; it was not optimized.
- The lower bounds are of order  $\sqrt{n}$ ; finite data suggest that  $f(n)$  is considerably larger, and we do not know its order of growth. The analysis of the one-skip family stops at  $15/\sqrt{21}$  (Remark 6.6); other patterns, several skips or other jump sizes change the case analysis and have not been explored systematically.
- The threshold  $n \geq 12096 \cdot 504^2$  in Theorem 1.3 is an artifact of the formal proof (Remark 6.5).
- Novelty is supported only by the searches described in `docs/LITERATURE.md` of the repository [21]; we have not read the full texts of [15] and [6] (of the latter only the abstract), and no expert in the area has been consulted.
- The proofs have been checked by machine and by AI systems; no independent human mathematician has checked them.

## STATEMENT ON THE USE OF AI TOOLS

This work was carried out with AI systems, and we describe that use in full. The research repository in which it was done was set up and directed by the author, who chose the problem, commissioned the runs described below and decided what to publish.

- The modulus-22 predecessor of Theorem 1.2 was found and formally proved on 9 September 2026 by an autonomous coding agent (OpenAI Codex, model identifier `gpt-6-astra`).
- Theorem 1.2, Theorem 1.3, the first two versions of the upper bound (constants  $\frac{125}{252}$  and  $\frac{797}{1620}$ ) and their formal proofs were produced on the night of 22–23 October 2026 by a multi-agent run of Claude models (Anthropic) coordinated by an AI agent; the coordinator’s log records that the agents ran on Claude Sonnet 5.5, and the trailer of one commit names Claude Opus 5.5.
- Theorem 1.1 (the five-band design and its formal proof) was produced on 23 October 2026 by a further Claude agent in the same repository; its commit trailer names Claude Sonnet 5.5.

- The audits (independent rebuilds, axiom audits, kernel replay, independent computations, literature searches and adversarial reviews), the written proofs in their present form and the text of this paper were produced on 23 October 2026 with Claude (Anthropic, model Claude Opus 5.5).

The records available to us (logs, commit history and the agents' reports) attribute the mathematics and the formal proofs to these AI systems and record no human mathematical contribution. The results have not been checked by an independent human mathematician or refereed; they are supported by the formal verification of Section 7. The author takes full responsibility for the content of this paper, including the accuracy of this statement, and for correcting or withdrawing any result that is found to be wrong.

#### ACKNOWLEDGEMENTS

We thank the maintainers of the Erdős problems database and the contributors to its forum, and the authors of the Formal Conjectures project, whose formal statement of Problem 357 we used.

#### REFERENCES

1. Adrian Beker, *On a problem of Erdős and Graham about consecutive sums in strictly increasing sequences*, Bull. Lond. Math. Soc. **56** (2024), no. 8, 2749–2759, arXiv:2311.10087.
2. Thomas F. Bloom, *Erdős Problem #1213*, <https://www.erdosproblems.com/1213>, Accessed 23 October 2026.
3. ———, *Erdős Problem #357*, <https://www.erdosproblems.com/357>, Accessed 23 October 2026.
4. ———, *Erdős Problem #874*, <https://www.erdosproblems.com/874>, Accessed 23 October 2026.
5. chy4pro, *erdos-357-distinct-block-sums*, <https://github.com/chy4pro/erdos-357-distinct-block-sums>, Table of  $f(n)$  for  $n \leq 53$  with witnesses; accessed 23 October 2026.
6. Don Coppersmith and Steven Phillips, *On a question of Erdős on subsequence sums*, SIAM J. Discrete Math. **9** (1996), no. 2, 173–177.
7. Leonardo de Moura and Sebastian Ullrich, *The Lean 4 theorem prover and programming language*, Automated Deduction – CADE 28, Lecture Notes in Comput. Sci., vol. 12699, Springer, Cham, 2021, pp. 625–635.
8. Jean-Marc Deshouillers and Gregory A. Freiman, *On an additive problem of Erdős and Straus, 1*, Israel J. Math. **92** (1995), no. 1–3, 33–43.
9. ———, *On an additive problem of Erdős and Straus, 2*, Astérisque **258** (1999), 141–148.
10. Erdős Problem a Day, *357: half the runs*, <https://erdosproblemaday.com/day/357-half-the-runs>, Erdős Problem a Day, 24 July 2026; accessed 23 October 2026.
11. Erdős Problems forum, *Discussion thread for Problem 357*, <https://www.erdosproblems.com/forum/thread/357>, Comments by D. Weisenberg (24 August 2025, 9 December 2025), Q. Tang (8 December 2025) and J. Pickhardt (31 August 2026); accessed 23 October 2026.
12. Paul Erdős, *Problems and results on combinatorial number theory III*, Number Theory Day (Proc. Conf., Rockefeller Univ., New York, 1976), Lecture Notes in Math., vol. 626, Springer, Berlin, 1977, pp. 43–72.
13. Paul Erdős and Ronald L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique, vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.
14. Google DeepMind and the Formal Conjectures contributors, *Formal Conjectures*, <https://github.com/google-deepmind/formal-conjectures>, File FormalConjectures/ErdosProblems/357.lean; accessed 23 October 2026.
15. N. Hegyvári, *On consecutive sums in sequences*, Acta Math. Hungar. **48** (1986), no. 1–2, 193–200.
16. Jakub Konieczny, *On consecutive sums in permutations*, J. Comb. **12** (2021), no. 3, 413–477, arXiv:1504.07156.
17. Lean FRO, *lean4checker*, <https://github.com/leanprover/lean4checker>, Tag v4.27.0.
18. C. Lenthall-Cleary, *Proof claim for Erdős Problem 357:  $f(n) \leq n/2 + O(n^{2/3})$* , Erdős problems forum, proof claims for Problem 357. A draft with the same bound, dated the same day, is at <https://github.com/chapinalc/chapinerdos>, Submitted 27 July 2026; not refereed; accessed 23 October 2026.

19. Paratelligent Research Agent and Jeff Pickhardt, *New bounds for sequences with distinct consecutive sums in Erdős Problem 357*, <https://paratelligent.com/research/papers/new-bounds-for-sequences-with-distinct-consecutive-sums-in-erds-1hwQT0M4>, 2026, Manuscript dated 14 July 2026, announced on the Erdős Problems forum on 31 August 2026; accessed 23 October 2026.
20. Bartłomiej Pawlik, *Sequence A364132*, The On-Line Encyclopedia of Integer Sequences, <https://oeis.org/A364132>, 2023, Terms  $a(14)$ – $a(22)$  by Jinyuan Wang; accessed 23 October 2026.
21. Chandler Snively, *Erdős Problem 357: Lean formalization, verification scripts and computations*, <https://github.com/dreadpiratechan/erds-357>, 2026, Release v1.0.
22. E. G. Straus, *On a problem in combinatorial number theory*, J. Math. Sci. **1** (1966), 77–80.
23. The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.

QUANA LABS

Email address: [lab@quanalabs.com](mailto:lab@quanalabs.com)

URL: <https://quanalabs.com>